# ON GROWTH, INSTABILITY, AND ASYMMETRIES

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# Innovation, Business Firms Growth, and Aggregate Fluctuations

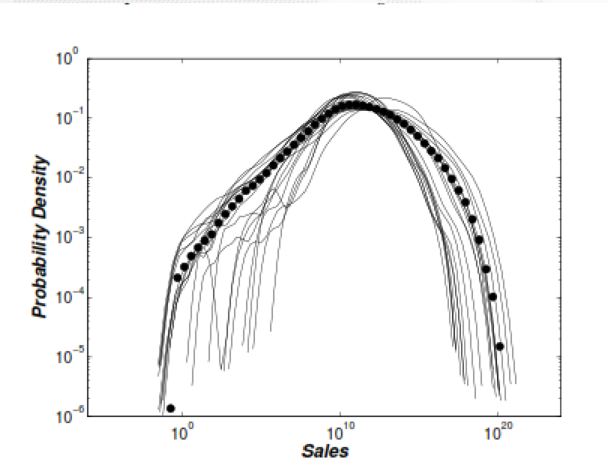
Growth and Volatility: Micro data availability and statistical regularities across domains

Innovation, turnover, and the composition of the economy: Micro level shocks diluted by aggregation or key elements to understand aggregate volatility?

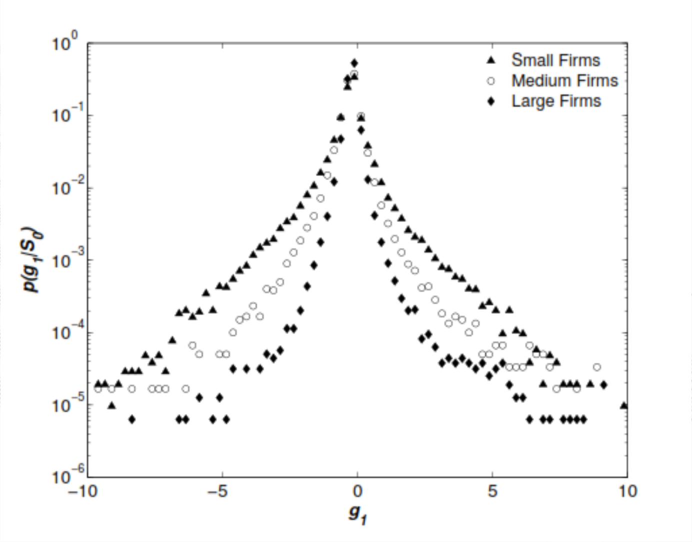
Growth and Inequality: Empirical findings, theoretical puzzles

# Empirical Evidences

The firm size distribution is skewed

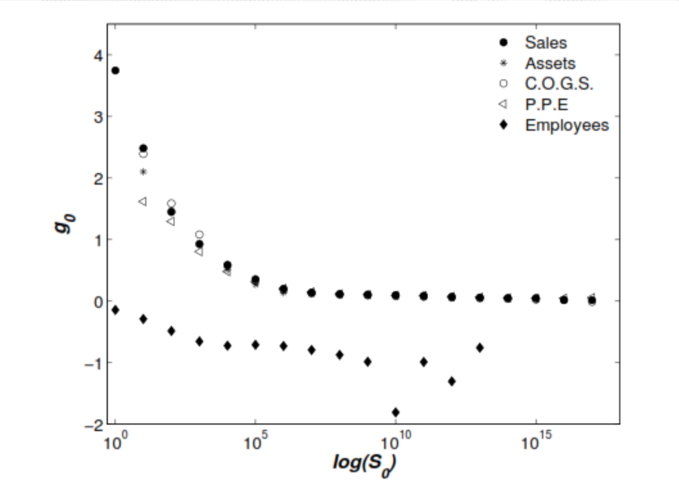



The growth rate distribution is «tent shaped»

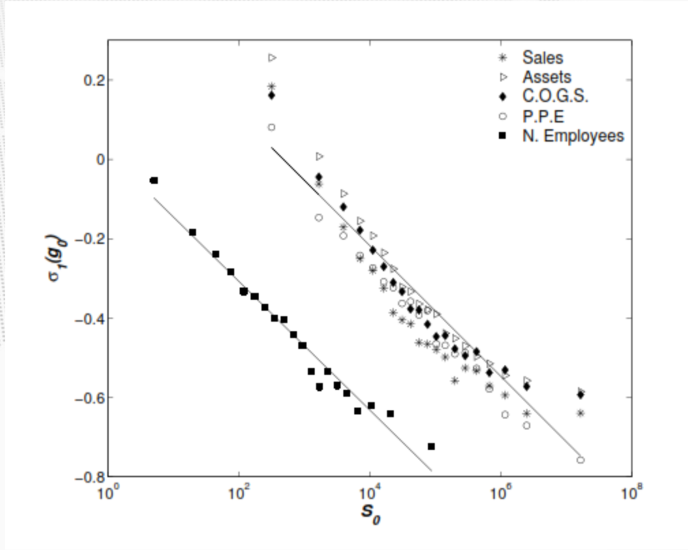


# Empirical Evidences

The relationship between size and average growth rate is negative




The variance of growth rates is systematically higher for smaller firms



# Innovation and Growth: A Stochastic Benchmark

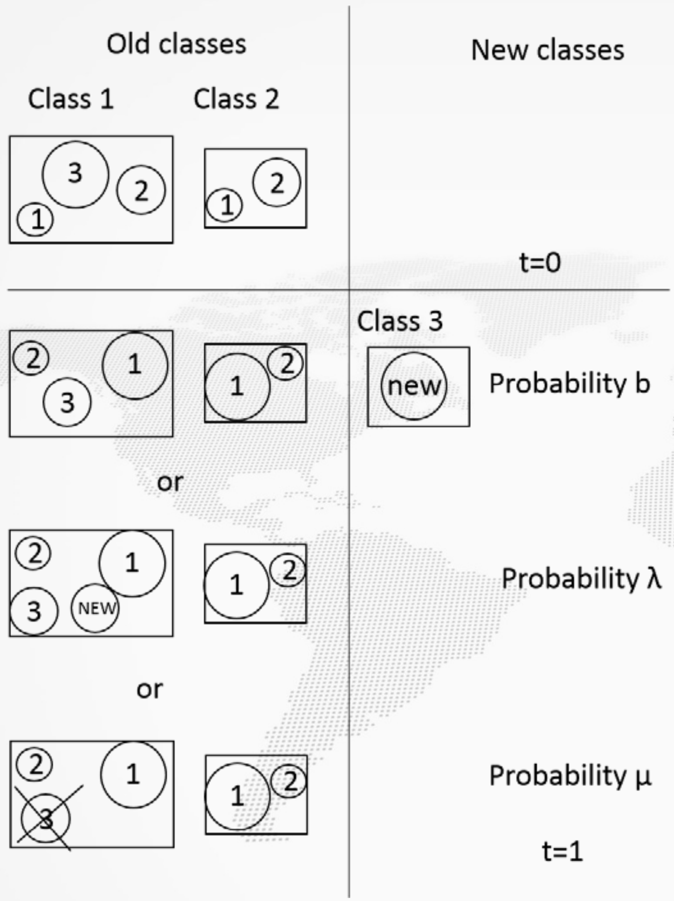



Fig 1. Schematic representation of the stochastic benchmark. At time t=0, there are N(0)=2 classes (□) and n(0)=5 units (○) Assumption (1). The area of each circle is proportional to the size ξ of the unit, and the size of each class is the sum of the areas of its constituent units (assumption 4) At time t=1 a new unit is created or deleted. With probability b the new unit is assigned to a new class (Class 3 in the figure) (Assumption 2). The size of the unit is taken from the distribution of the existing units (assumption 5). With probability λ the new unit is assigned to an existing class with probability proportional to the number of units in the class (assumption 2). In this example, a new unit is assigned to class 1 with probability 3/5 or to class 2 with probability 2/5. With probability μ, a randomly selected unit is deleted.

# PREDICTIONS

1. The distribution of the number of products per firm, P(K) is Pareto, with an exponential cut-off (entry-exit).
2. The size distribution of products is Lognormal
3. The size distribution of firms is Pareto
4. The growth distribution of firms is tent shaped with power law wings
5. The relationship between size and average growth rates is decreasing
6. The relationship between size and variance of growth rates has a power law behavior = $\sigma(S) \sim S \exp(-\beta(S))$ with $0<\beta<1/2$

Predictions tested at different level of aggregation:

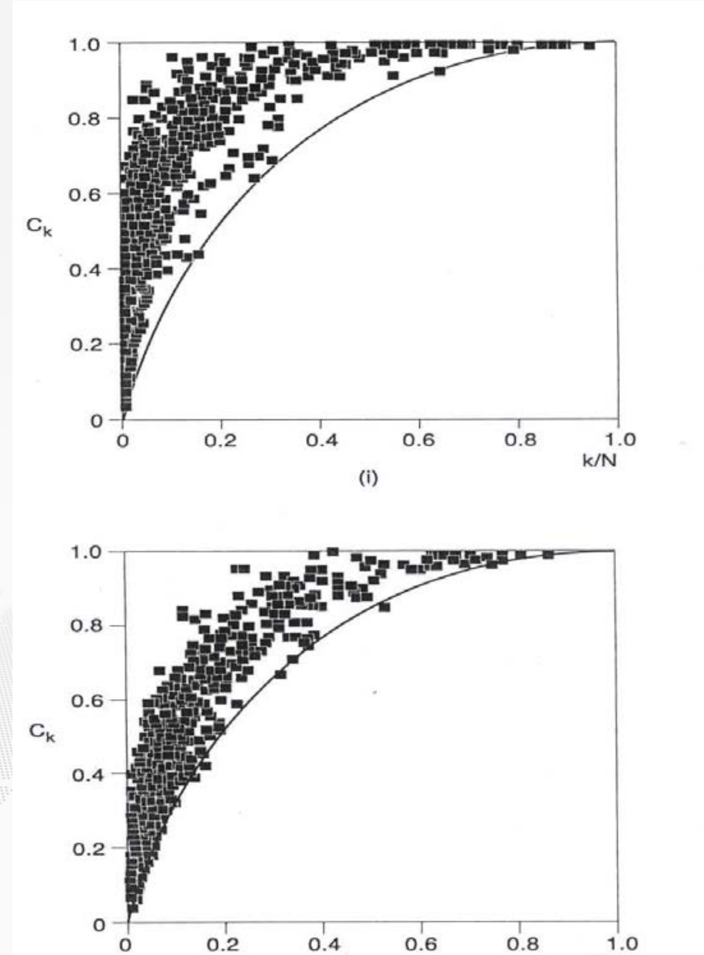
- The GDP of 195 countries, from 1960 to 2011 (World Bank);
- U.S. manufacturing publicity-traded firms from 1951 to 2012;
- A unique, proprietary, data set with micro data at the product and at the firm lev

John Sutton: It is sufficient to assume that the absolute growth rate is non decreasing in firm size, to obtain that in the limit where the number of opportunities becomes large, we obtain a size distribution, which
features a certain minimum degree of inequality in the size distribution of firms.

Specifically, it leads to the prediction that the Lorenz curve must lie farther from the diagonal than a limiting 'reference curve', which is defined by the relationship

$$C_k \geq \frac{k}{N}\,(1\text{-}\ln \frac{k}{N}\,)$$

where $C_k$ is the k-firm concentration ratio (the fraction of all opportunities shared by the k largest firms in the industry, and N is the number of firms. (The case of equal sizes would correspond to $C_k$ = k/N, and here the Lorenz curve lies on the diagonal.)




The top panel shows the scatter diagram of Ck against k/N for pooled data
(k=4,8 and 20) for the United States 1987, at the four-digit level. The Lorenz curve shown on these figures is the reference curve
The bottom panel shows data for Germany , 1990 (k=3,6,10 and 25) (Sutton, 2009).

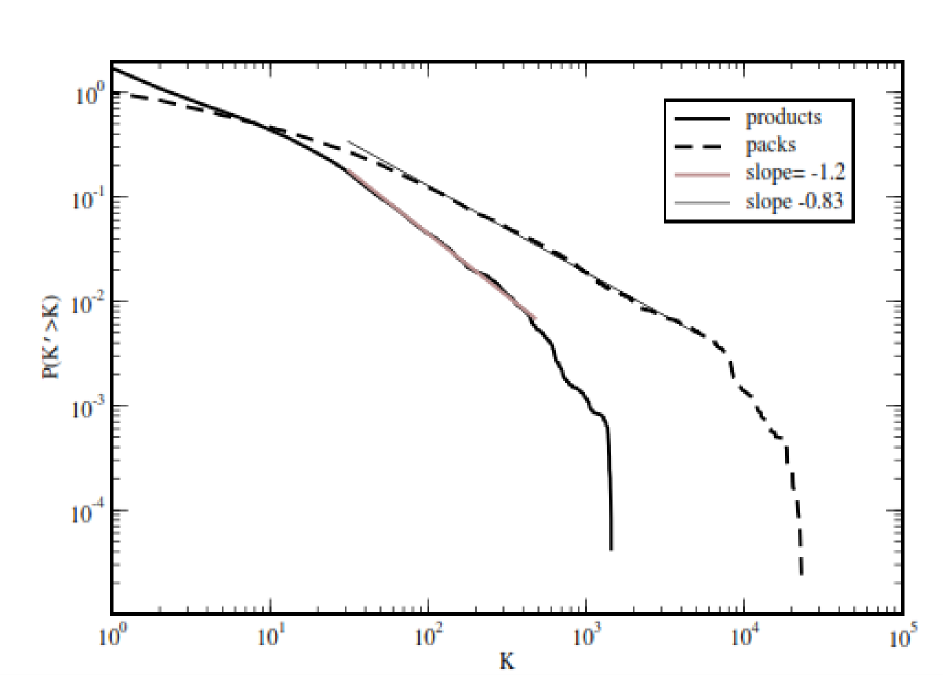
## MICRO DATA ON CORPORATE PRODUCT PORTFOLIOS

Micro Data: Sales figures of 189,303 products commercialized by 4,921 companies in 19 countries from 1998 to 2008. Universe of products and companies. Decomposition of business firms sales figures into individual components (n, S(i)) Updated version of the db ready to go.

Each product is classified in one of 574 independent submarkets.

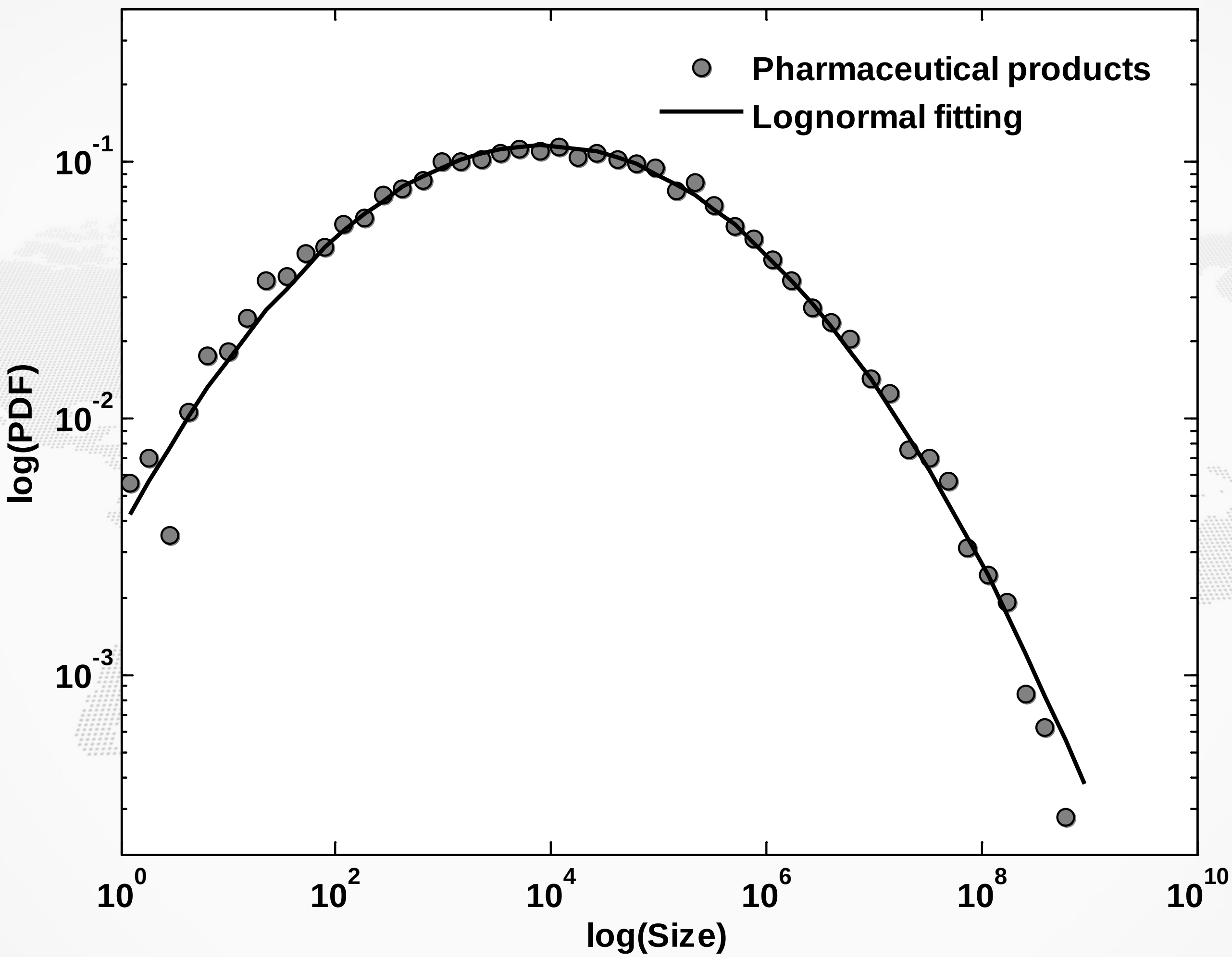
Entry and Exit of products and firms.

Distribution of the number of products per firm, P(K): Pareto distribution with an exponential cut-off (entry-exit).



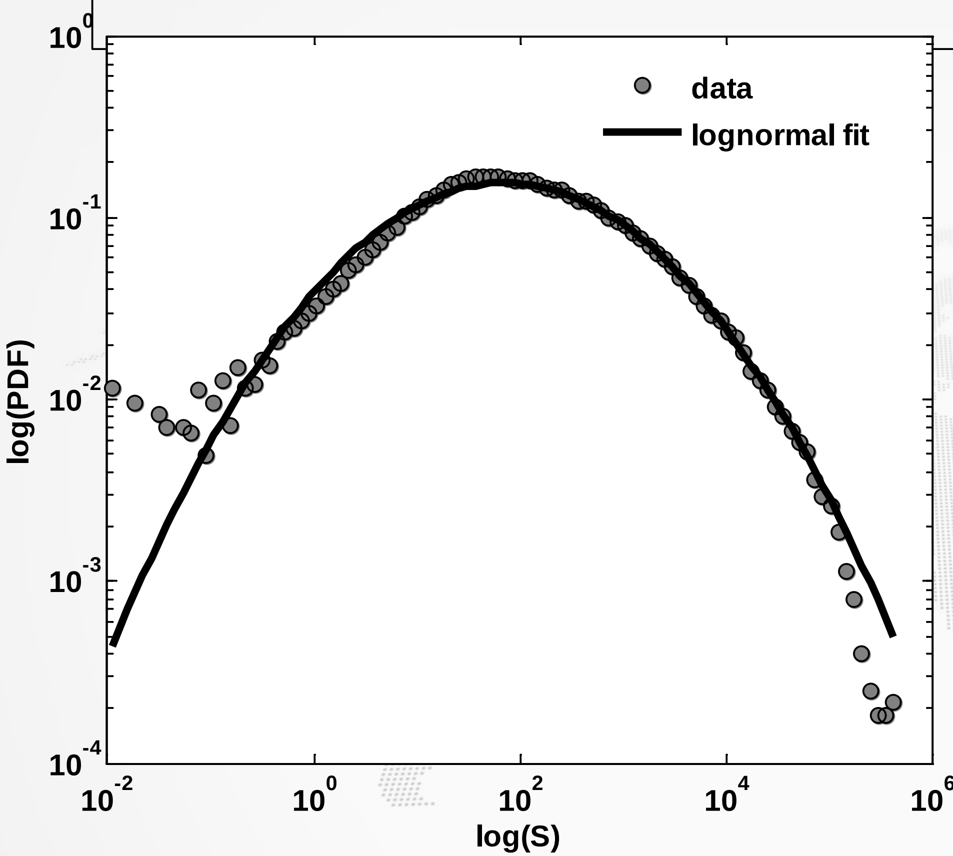

Cumulative distribution of the number of products in all firms.

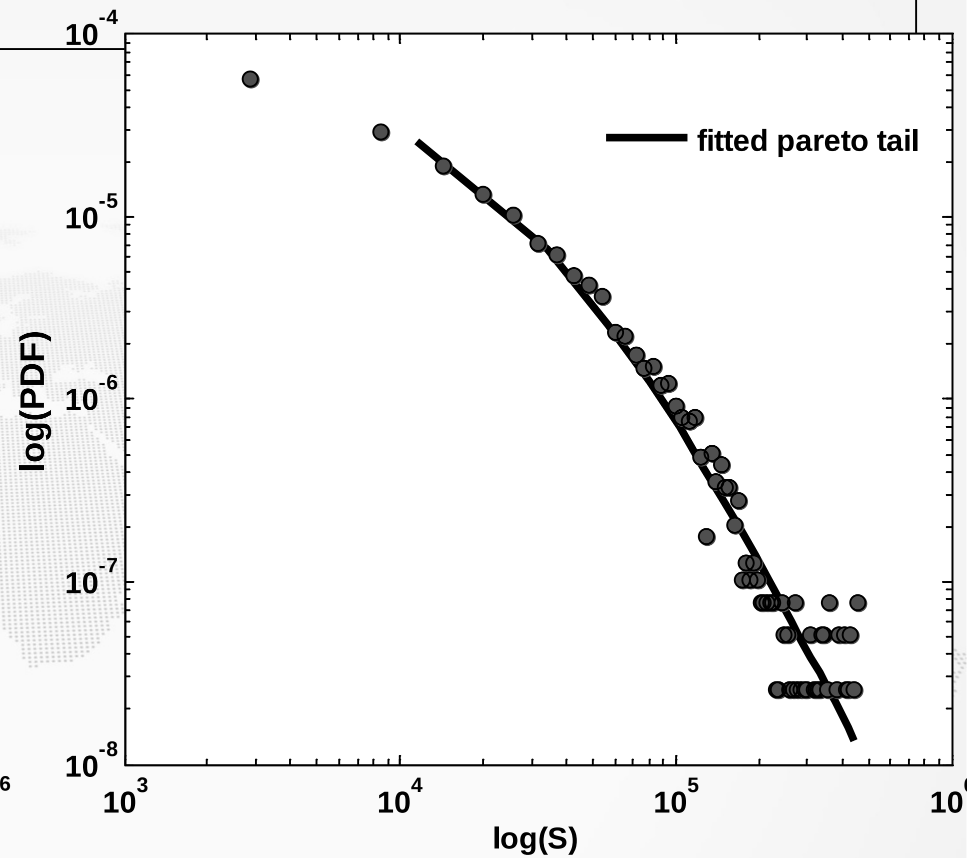
# Product size distrinbution in the US market: Lognormal



# FIRM SIZE DISTRIBUTION

Size distribution: Aggregation and the emergence of a Pareto tail.



# Innovation and Growth at Different Levels of Aggregation

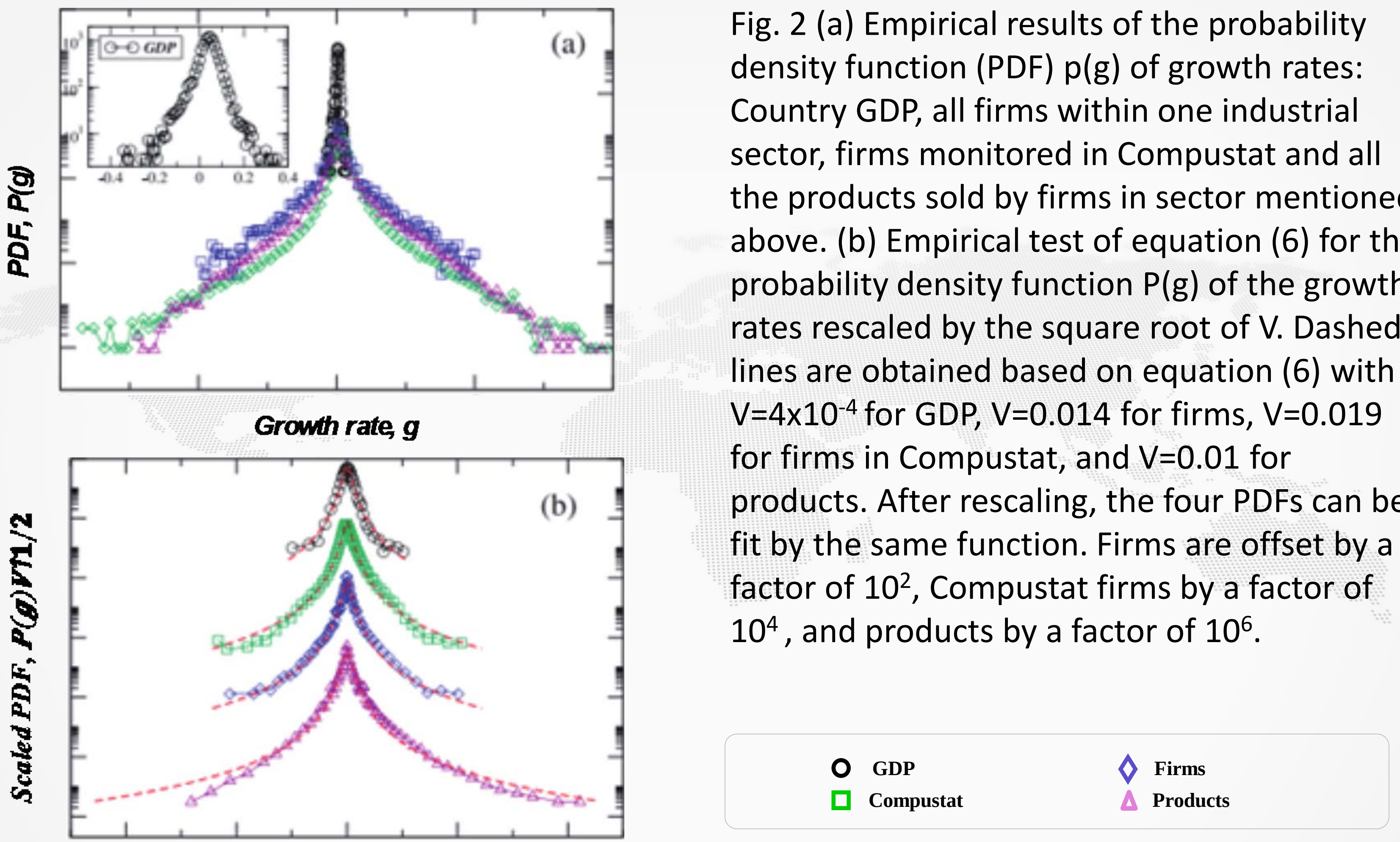
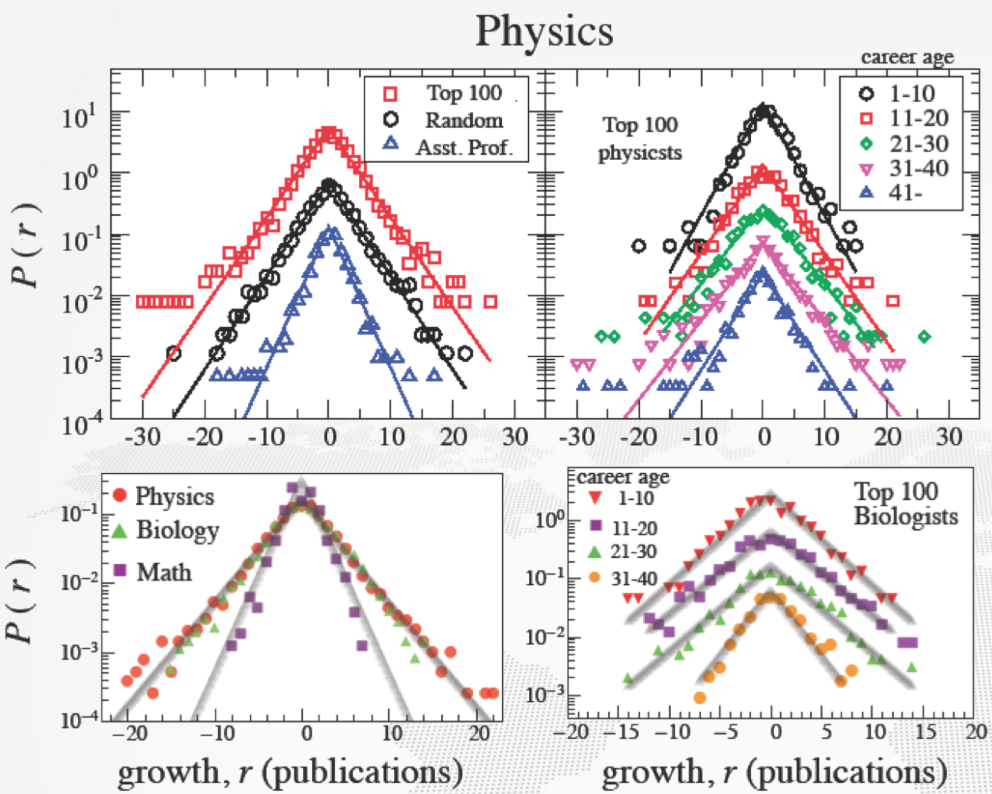



Fig. 2 (a) Empirical results of the probability density function (PDF) p(g) of growth rates: Country GDP, all firms within one industrial sector, firms monitored in Compustat and all the products sold by firms in sector mentioned above. (b) Empirical test of equation (6) for the probability density function P(g) of the growth rates rescaled by the square root of V. Dashed lines are obtained based on equation (6) with $V=4x10^{-4}$ for GDP, V=0.014 for firms, V=0.019 for firms in Compustat, and V=0.01 for products. After rescaling, the four PDFs can be fit by the same function. Firms are offset by a factor of $10^2$, Compustat firms by a factor of $10^4$ , and products by a factor of $10^6$.

# A Pervasive Feature. From Scientific Output, Trade Flows…



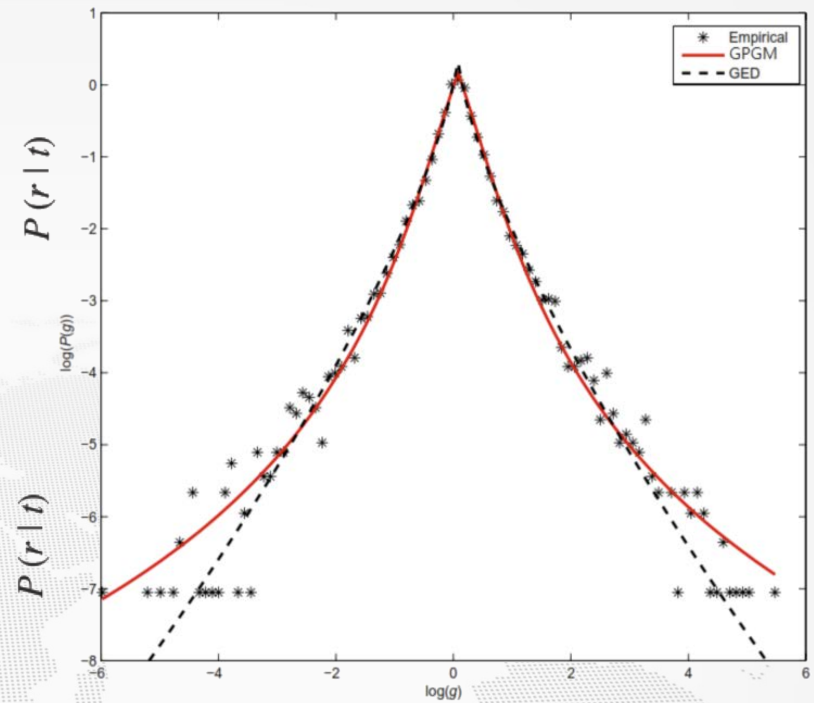



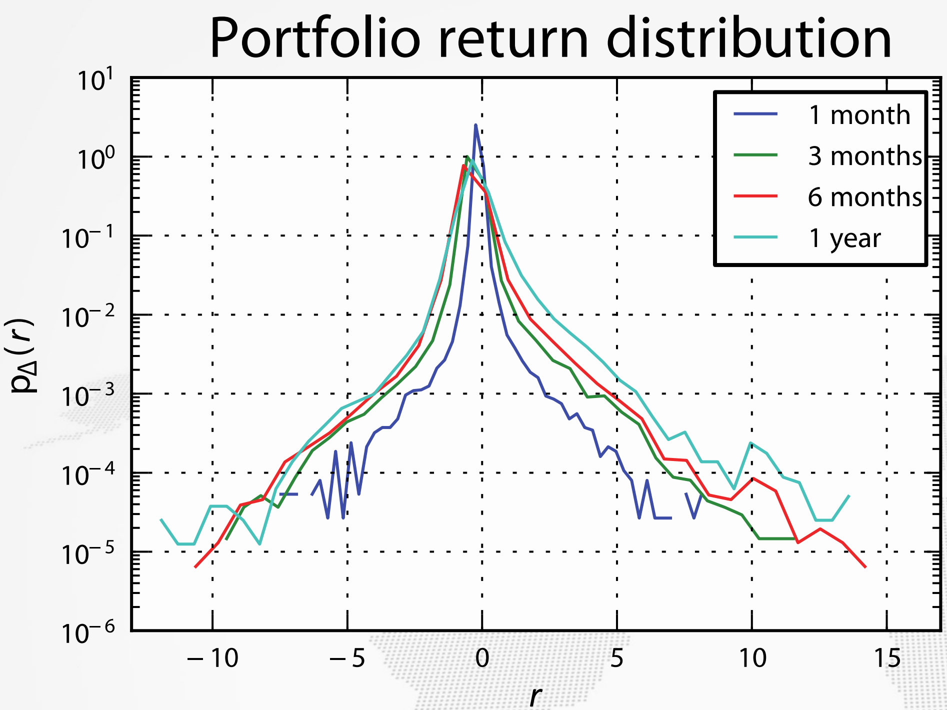

Fig 6. Distributions of the publication growth, defined as the change in the number of publications per year for 3 cohorts of top scientists, also disaggregated at career age.

A. M. Petersen, M. Riccaboni, H. E. Stanley, F. Pammolli (2012) “Persistence and Uncertainty in the Academic Career”. *Proceedings of the National Academy of Sciences USA,* 109: 5213 - 5218

Fig 7. Distribution P(g) of the growth rate g of aggregate trade flows

Riccaboni M, Schiavo S. (2010), ‘Structure and growth of weighted networks’, *New Journal of Physics. 12-023003*.

# .... To Investment >Funds

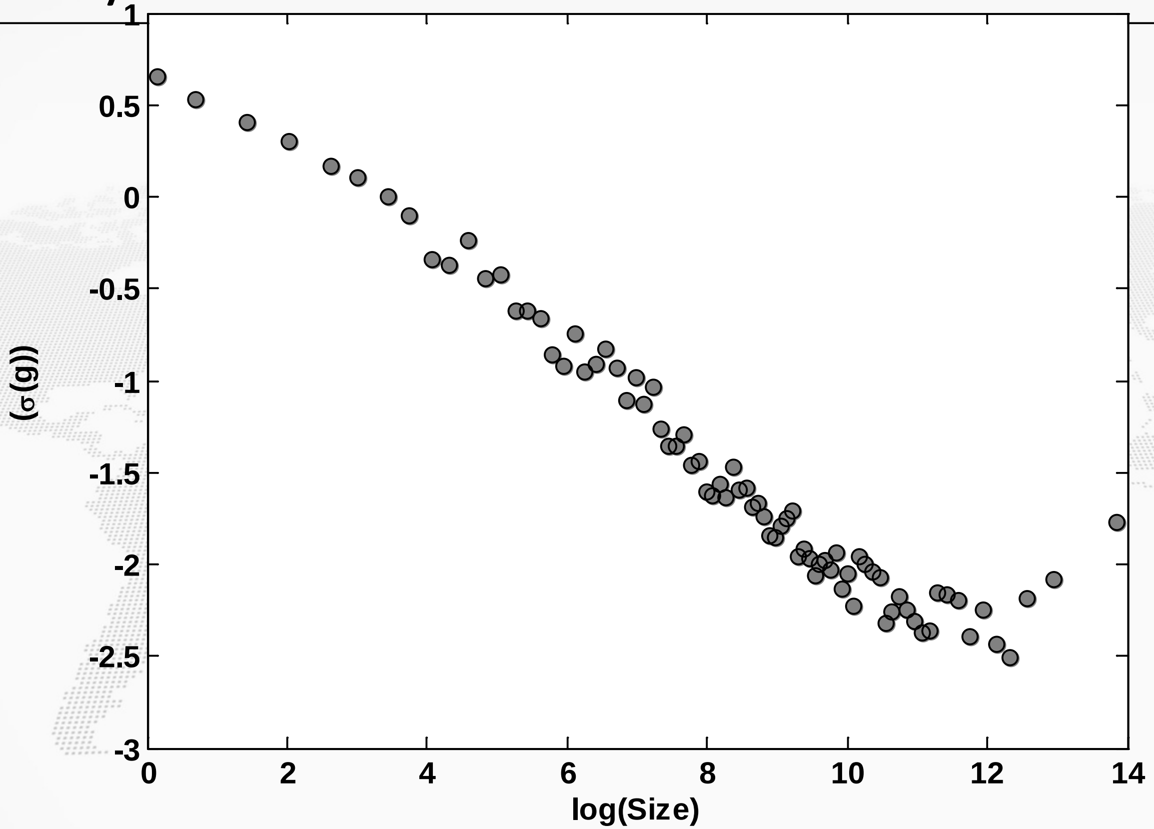



**Probability density function p(*r*) of the logarithmic growth of funds portfolios**: **variations computed for time horizons ranging from 30 days to one year.**
Data: CRSP on US mutual funds' portfolio holdings, November 2011 - September 2012, for a total of 19493 portfolios investing in a total of 576598 different assets and total investments value ranging from 371 billions to 81 trillions.

Source: Pammolli, Buldyrev, Flori, et al., forthcoming

# A GENERALIZED MODEL OF PROPOTIONAL GROWTH

The GMPG predicts a negative relationship between the size of firms and the mean growth rate. The negative relationship is confirmed by the data.




Relationship between the logarithm of size (S) and its mean growth rate (g). The growth rate is calculated as $g = \frac{S_t - S_{t-1}}{S_{t-1}}$

# Size, Diversification, and Instability: From Products to GDP

*In case of skewed size distributions, the size-variance relationship scales with the share of the largest unit*

The size-variance relationship crucially depends on the partition of firm (country) size across constituent components. If firms have P(K) units and $V_\eta=0$, for the Law of Large Numbers, $\sigma(K) \approx K^\beta$, where $\beta = 1/2$. On the contrary, if each firms consists of a single unit only and $V_\eta > 0$, $\beta = 0$. When both mechanisms are at work, the speed of the crossover depends on the skewness of P(K). At one extreme, if all entities have the same number of units, $\beta=0$ and there is no crossover. On the contrary, if P(K) is power-law distributed, for a wide range of empirically plausible $V_\eta$ $\beta$ is far from 1/2 and statistically different from zero. The size-variance relationship is not a true power law with a single well-defined exponent $\beta$, but undergoes a slow cross over from $\beta=0$ for $S \rightarrow 0$, to $\beta=1/2$ for $S \rightarrow \infty$.

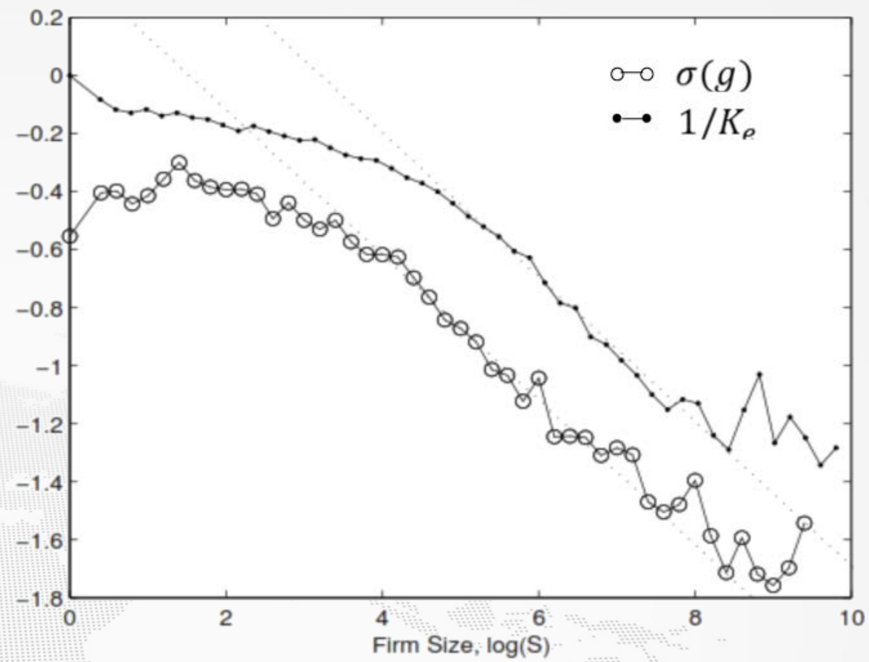
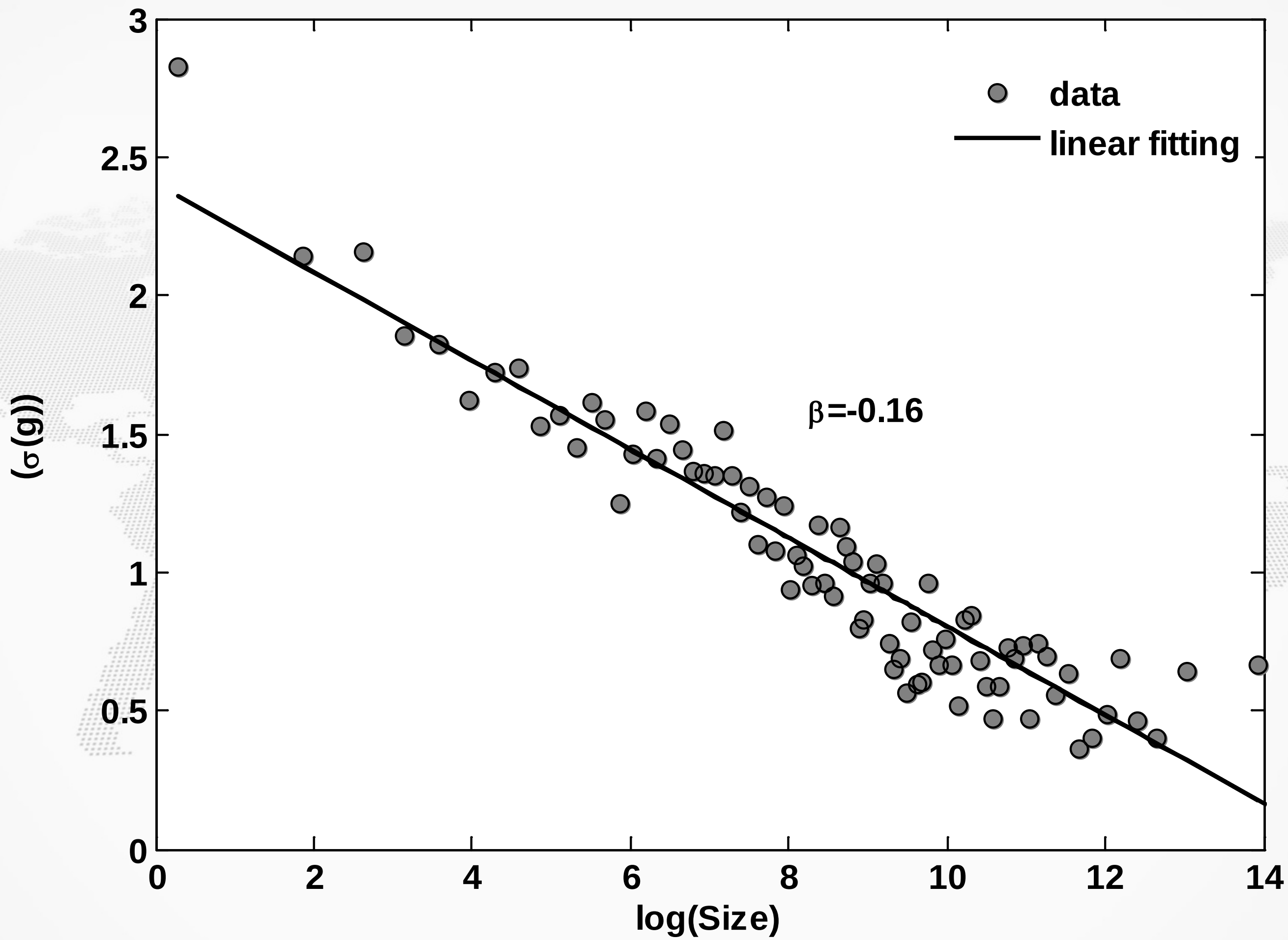



*Fig. 4* Size-Variance relationship. The standard deviation of firm growth rates (σ) (circles), and the share of the largest products ($1/K_e$) (squares) versus the size of the firms (*S*). For $S < S1 = \mu_\xi \approx 3.44$, $\beta \approx 0$. For $S > S1$ $\beta$ increases but never reaches 1/2 because of the slow growth of the number of products ($K_e$). The flattening of the upper tail is due to some large companies with unusually large products.

The relationship between size and variance of growth rates has a power law behavior = $\sigma(S) \sim S \exp(-\beta(S))$ with $0<\beta<1/2$. In our data, $\beta=-0.16$.



# Innovation, Instability, and Growth: Discussion

*Human Know How, Capabilities, and the Wealth of Nations*. Entry of new business opportunities/new technologies/new firms is a key element of known growth distributions..

*Animal Spirits', Firms, Market Structure, and 'the State of the Economy'*: Firm level and sector specific dynamics are an important part of business cycle aggregate fluctuations (even without hub-like general purpose inputs). Impact of GPIP

*Diversification and Country Size reduce Volatility*, but less than expected. *Rare and extreme events are not outliers. Undoing of the law-of-large-numbers argument*

*Entry, Growth, and Unevenness* as instantiations of the same process. Human capital accumulation, industrial and social mobility.

Scarce capabilities, clustering, trade, prices of production factors: low frequency processes and global, high frequency, amplifying factors: *A Supranational Dimension for Stabilization and Investment?*

# Supplementary Materials

# The Generalized Proportional Growth Model (GPGM)

The GPGM model includes the Gibrat proportional growth and the Simon preferential attachment:

— the number of units in a firm grows in proportion to its existing number of units (the Simon growth process);

— the size of each unit grows in proportion to its size, independently of other units (the Gibrat growth process).

Two key sets of assumption in the model are that the number of units in a class grows in proportion to the existing number of units (1-4) and the size of each units fluctuates in proportion to its size (5-7)

# The Generalized Proportional Growth Model (GPGM)

(1) At time $t$ the system consists of $N(t)$ firms. Each firm $i$ consists of $K_i(t)$ units. We characterize the system by the number of firms, $N_k(t)$, consisting of exactly $k$ units. By definition

$$N(t) = \sum_{k=0}^{\infty} N_k(t).$$

The total number of units in the system $n(t)$ is

$$n(t) = \sum_{k=0}^{\infty} kN_k(t) \equiv \langle K(t) \rangle N(t),$$

where $\langle K(t) \rangle$ is the average number of units in the firm. We assume that at time $t = 0$ there are $N_k(0)$ firms consisting of $k$ units. We denote the initial number of firms and units as $N(0) \equiv N_0$ and $n(0) \equiv n_0$, respectively. Accordingly, we introduce initial average number of units in the firm

$$\langle k \rangle = n_0/N_0 = \langle K(0) \rangle.$$

We introduce $N_0(t)$ to account for the currently inactive firms, those that have lost all their units. We define the initial distribution of firm sizes as $P_k^o = N_k(0)/N_0$

# The Generalized Proportional Growth Model (GPGM)

(2) At each time interval $\Delta t$, a number of new units $\Delta_\lambda n$ is created in proportion to the current size of the economy measured in the total number of units: $\Delta_\lambda n = \lambda n(t)\Delta t$, where $\lambda$ is the growth rate. These units are distributed among existing firms with probability $p_i$, which is proportional to the size of firm $i$: $p_i = K_i(t)/n(t)$.

(3) At each time step, any unit can be deleted with probability $\mu$. Thus the number of units deleted during time interval $\Delta t$ is $\Delta_\mu n = \mu n(t)\Delta t$. The probability that a deleted unit belongs to the firm $i$ is $p_i = K_i(t)/n(t)$.

# The Generalized Proportional Growth Model (GPGM)

(4) At each time interval $\Delta t$, a number of new firms $\Delta_\nu N = \nu' n(t)\Delta t$ is created, where $\nu'$ is the birth rate of new firms. We assume that a new firm has $k$ units with probability $P'_k$. Thus the total number of units added to new firms is $\Delta_\nu n = \nu n(t)\Delta t$, where

$$\nu \equiv \nu' \sum_k P'_k k = \nu' \langle k \rangle'$$

and $\langle k \rangle'$ is the average number of units in the new firms.

# The Generalized Proportional Growth Model (GPGM)

Based on Assumptions (1-4) the number of units $n(t)$ obeys a differential equation

$$\frac{dn}{dt} = (\lambda - \mu + \nu)n(t)$$

from where

$$n(t) = n_0 e^{(\nu+\lambda-\mu)t}$$

and

$$N(t) = \frac{\nu'}{\nu + \lambda - \mu}(n(t) - n_0) + N_0$$

# The Generalized Proportional Growth Model (GPGM)

(5) At time $t$, each firm $i$ has $K_i(t)$ units of size $\xi_j(t)$, $j = 1, 2, ...K_i(t)$ where $\xi_j > 0$ are independent random variables taken from the distribution $P_\xi$. We assume that $\mathrm{E}[\ln \xi_i(t)] \equiv m_\xi$ and $\mathrm{Var}[\ln \xi_i(t)] = \mathrm{E}[(\ln \xi_i)^2] - m_\xi^2 \equiv V_\xi$, where $\mathrm{E}[x]$ and $\mathrm{Var}[x]$ are respectively mathematical expectation and variance of a random variable $x$. The size of a firm is defined to be $S_i(t) \equiv \sum_{j=1}^{K_i(t)} \xi_j(t)$.

(6) At time $t + 1$, the size of each unit is decreased or increased by a random factor $\eta_j(t) > 0$ so that

$$\xi_j(t+1) = \xi_i(t)\,\eta_i(t).$$

We assume that $\eta_j(t)$, the growth factor of unit $j$, is a random variable taken from a given probability distribution $P_\eta$. It is assumed that $\mathrm{E}\ln \eta_i(t) \equiv m_\eta$ and $\mathrm{Var}[\ln \eta_i(t)] = \mathrm{E}[(\ln \eta_i)^2] - m_\eta^2 \equiv V_\eta$. Note that $\eta_j$ is independent of $\xi_j, K_i$ and all other random variables characterizing the firm.

# The Generalized Proportional Growth Model (GPGM)

Based on these assumption we derive the predictions for:

- the size distribution of firms;
- the distribution of firm growth rates;
- the size-mean growth rate relationship
- the size-variance relationship.

# The Generalized Proportional Growth Model (GPGM)

In general for a given distribution $P_\xi(\xi)$ the distribution of the firm size is given by: $$P(S) = \sum_{K=1}^{\infty} P(S|K) P_K,$$

Where: $P(S|K) = P_\xi^{(K)}(S),$ is the sum of (i.e. convolution of of K distribution $P_\xi(\xi)$) .

we have convolution of lognormals → the convergence to a Gaussian depends on $V_\xi$ and K.
when $V_\xi$ is reasonably high ($V_\xi$ = 10 in our empirical observed data base) if K<100 the distribution resemble the original lognormal.

$P_k$→ distribution of number of units within classes → with our assumption, con be proved that the $P_k$ is Pareto with exponential cut off .

P(S)→ is a lognormal distribution with a Pareto tail.

# The Generalized Proportional Growth Model (GPGM)

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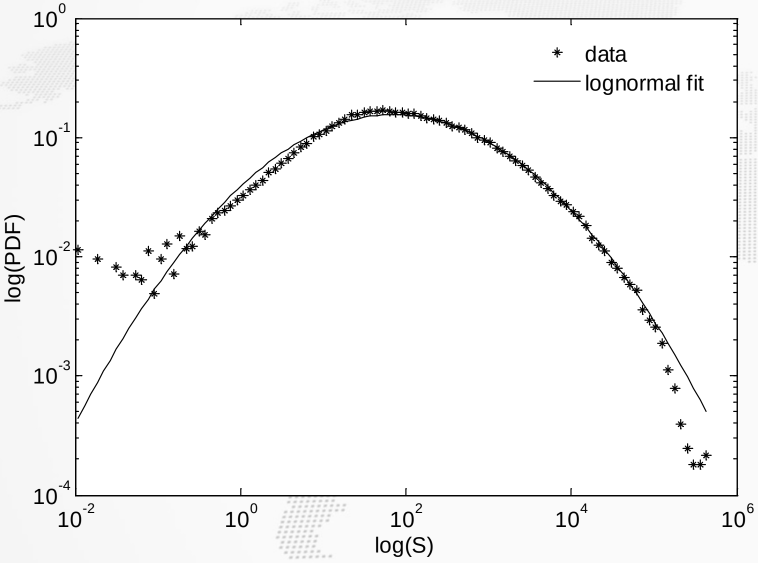
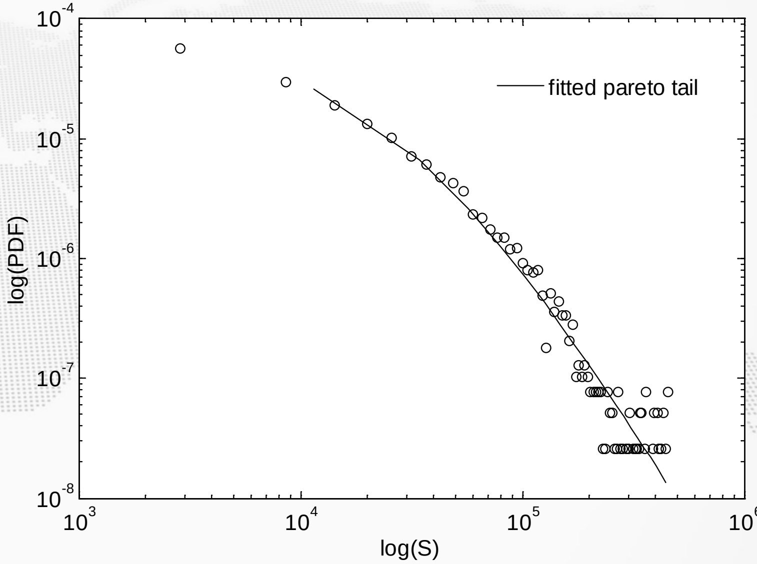
# The Generalized Proportional Growth Model (GPGM)

It can be proved that if $\nu>0$, i.e. system with new entries, the distribution of the firm sizes converge to a lognormal with a Pareto right tail.



# The Generalized Proportional Growth Model (GPGM)

The distribution of growth rates is given by:

$$P_r(r) \equiv \sum_{K=1}^{\infty} P_K P_r(r|K),$$

where $P_k$ is the distribution of the number of the units in each firm and $P_r(r|K)$ is the conditional distribution of growth rates of firms with a given number of units determined by the distribution $P_\xi(\xi)$ and $P_\eta(\eta)$.
We will assume that both $P_\xi(\xi)$ and $P_\eta(\eta)$ are lognormal distributions (as it follows from the Gibrat growth process):

$$P_\xi(\xi) = \frac{1}{\xi\sqrt{2\pi V_\xi}} \mathrm{e}^{-\frac{(\ln\xi - m_\xi)^2}{2MV_\xi}},$$

$$P_\eta(\eta) = \frac{1}{\eta\sqrt{2\pi V_\eta}} \mathrm{e}^{-\frac{(\ln\eta - m_\eta)^2}{2MV_\eta}},$$

# The Generalized Proportional Growth Model (GPGM)

In this case is not possible to obtain a closed form for P(r|K) and for its mean and variance.
The exact solution exist only in the limiting cases K→0 and K→∞. On the basis of the central theorem, in the limit of very large K, P(r|K) converge to a Gaussian:

$$P_r(r|K) = \frac{\sqrt{K}}{\sqrt{2\pi V_r}} \exp\left(-\frac{(r - m_r)^2 K}{2V_r}\right)$$

Where the mean growth rate is: $m_r = m_\eta + V_\eta/2 + \ln(1 + \lambda - \mu),$

and the normalized variance is:

$$V_r = \frac{(1 + \lambda - \mu)\exp(V_\xi)[\exp(V_\eta) - 1] + (\lambda + \mu)\exp(V_\xi)}{(1 + \lambda - \mu)^2} + \frac{(\lambda^2 - \mu^2)[\exp(V_\xi) - 1]}{(1 + \lambda - \mu)^2}.$$

# The Generalized Proportional Growth Model (GPGM)

Assuming that $P_k$ is exponential with $\langle K \rangle = k$ and replacing the summation with the integration (to find a closed form), we obtain:

$$P_r(r) \approx \frac{1}{\sqrt{2\pi V_r}} \int_0^{\infty} \frac{1}{\kappa(t)} \exp\left(\frac{-K}{\kappa(t)}\right) \exp\left(-\frac{(r-m_r)^2 K}{2 V_r}\right) \sqrt{K}\, dK,$$

$$= \frac{\sqrt{\kappa(t)}}{2\sqrt{2 V_r}} \left(1 + \frac{\kappa(t)}{2V_r}(r-m_r)^2\right)^{-\frac{3}{2}},$$

This distribution is a tent shape distribution that asymptotically decay at $1/r^3$.

# The Generalized Proportional Growth Model (GPGM)

PROBLEM: with the assumption that the growth rates $\ln(\eta)$ for the individual units is lognormal the GPGM predicts a growth rate distribution at the firm level that is tent shape only in the case when the distribution of number of units in the classes is not strongly dominated by the classes with few units (i.e exponential distribution).
In other cases the growth distribution does not developed tent-shape wings, because the behavior for large $r$ is dominated by the distribution of the growth rates $\ln(\eta)$ that we assume to be gaussian.

Empirical investigations suggest that this assumption is not correct, at least for the growth rate of products of the firms, which follows a tent-shape distribution.

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We derived a two-step GPGM model:
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We derived a two-step GPGM model:

-the most elementary units (that are not observable) have lognormal distribution of $P_\xi(\xi)$ and $P_\eta(\eta)$;
-the first level of aggregation (i.e.products of firms) consist of L elementary units with geometric distribution $P_1(L)$;
-the second level of aggregation (i.e. firms) consist of M of these compounds units with Pareto distribution of $P_2(M)$.

# The Generalized Proportional Growth Model (GPGM)

Once we obtain $P_r(r|K)$ and $P_k$ we can substitute in $$P_r(r) \equiv \sum_{K=1}^{\infty} P_K P_r(r|K),$$

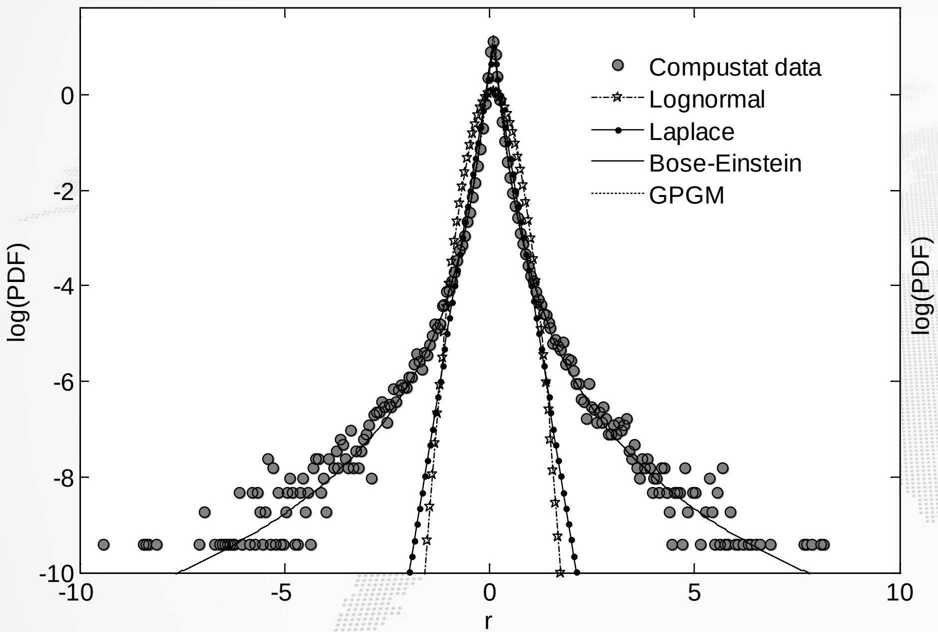
Without the replacement of summation by integration, we obtain:

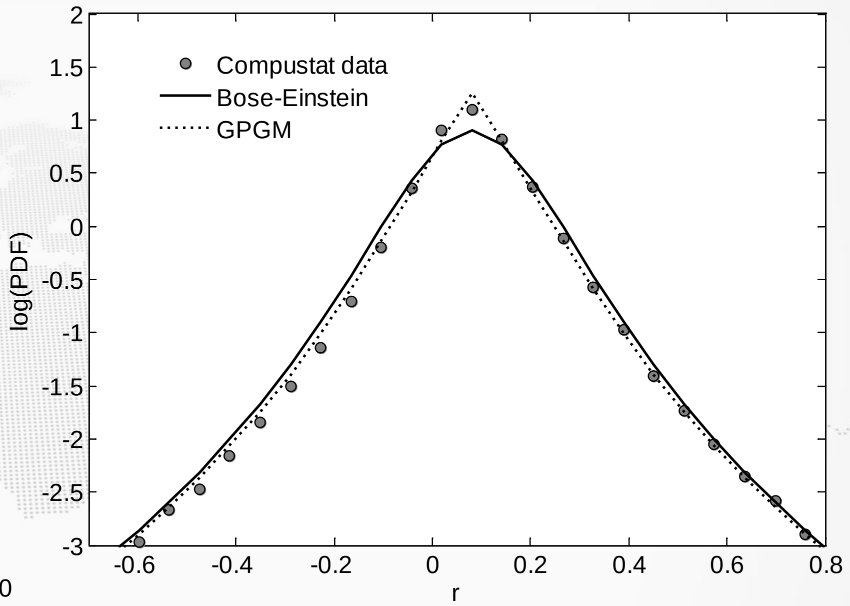
$$P(r) = \frac{1}{\sqrt{2\pi V}}(\kappa \mathrm{Bs}_0[\exp\left(\frac{r^2}{2V}\right)] - (\kappa - 1)\mathrm{Bs}_0[[\theta \exp\left(\frac{r^2}{2V}\right)] - \mathrm{Li}_{1/2}[\theta \exp\left(\frac{r^2}{2V}\right)]),$$

where : $\theta = 1 - 1/\kappa,$ $$\mathrm{Li}_{1/2}(x) = \sum_{k=1} \frac{x^k}{\sqrt{k}}$$ is the poly-logarithm and

$$\mathrm{Bs}_0(x) = \sum_{k=1} \frac{x^k}{(k+1)\sqrt{k}}$$ is the integral of the poly-logarithm.

# The Generalized Proportional Growth Model (GPGM)



# The Generalized Proportional Growth Model (GPGM)

Statistical tests of goodness of fitting (Kolmogorov-Smirnov and Anderson-Darling) confirm that the GPGM model outperforms the other model.

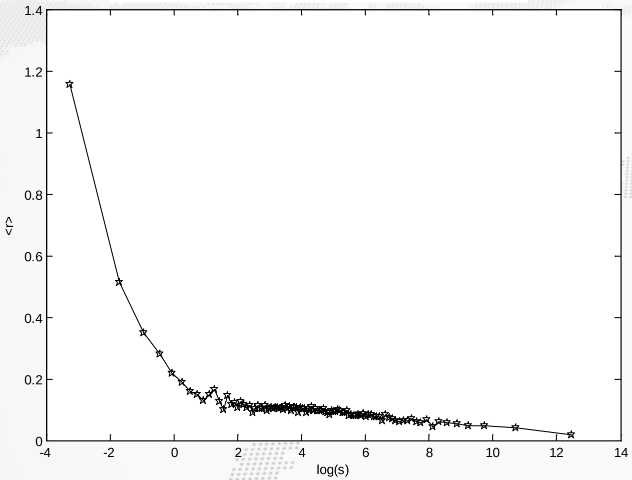
Maximun Likelihood Estimates (MLE) of the growth rate distribution.

| Distribution | $\mu$ | $\sigma$ | $\frac{k}{2V_r}$ | KS | AD |
|---|---|---|---|---|---|
| Gaussian | 0.0844 | 0.3702 | – | 17.1173 | 2.29E+79 |
| Laplace | 0.0844 | 0.1854 | – | 5.0453 | 1.10E+07 |
| Bose-Einstein | – | – | 24.5 | 3.7989 | 0.0837 |
| GPGM | – | – | 12.25 | 1.6734 | 0.0343 |

# The Generalized Proportional Growth Model (GPGM)

As regards to the relation between size and average growth rate the GPGM model we must stressed that:

$$\langle r \rangle = \langle \ln(S_i(t+1)/S_i(t)) \rangle.$$

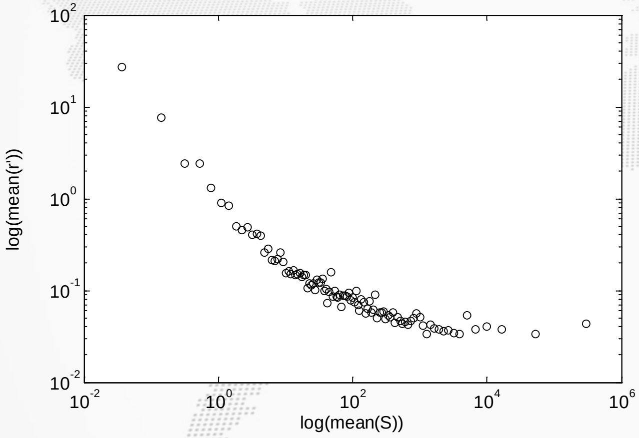



This definition excludes from the average the firms that lost all their products and hence $S_{t+1} = 0$. Accordingly it creates a positive bias for small firms consisting of a unit. For firms consisting on few units it creates a negative bias due to the asymmetry of the *ln(x)* function (which diverge to →∞ for x→ 0, but slowly increase for x→1. Only for very large firms, consisting of many independent units, $x = \frac{S_{t+1}}{S_t} \approx 1$ and the asymmetry of *ln(x)* becomes negligible.

# The Generalized Proportional Growth Model (GPGM)

As regards to the relation between size and average growth rate the GPGM model :

$$\langle r' \rangle = \langle S_i(t+1)/S_i(t) - 1 \rangle.$$




In the light of our assumptions the average growth rate is:

$$< r' >= \lambda[exp(m_\xi + V_\xi/2]/S - \mu$$

# The Generalized Proportional Growth Model (GPGM)